ON THE REDUCTION OF CERTAIN DYNAMIC PROGRAMMING PROBLEMS FOR NONLINEAR SYSTEMS TO TRANSCENDENTAL EQUATIONS

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In [1], concerning questions of dynamic programming theory related to the realization of a chosen strategy for control of motion, we studied the problem of the selection of controlling forces which would ensure the realization of the law of motion, given in a phase space (or subspace), of a nonlinear control system.

A preliminary requirement for the determination of the controlling forces in the problem being considered, is the solution, obtained in [1], of a system of nonlinear integral equations of the following form:

$$z_j(t) = \Gamma_j(t) - \sum_{i=0}^{m-1} \sum_{l=1}^{n} \chi_{ji}(t) \int_{t_0}^{t_1} \Xi_{il}(t_1, \tau)\, \psi_l(z_1(\tau), \ldots, z_r(\tau), \tau)\, d\tau +$$

$$+ \sum_{l=1}^{n} \int_{t_0}^{t} W_{jl}(t, \tau)\, \psi_l(z_1(\tau), \ldots, z_r(\tau), \tau)\, d\tau \quad (t_0 \leqslant t \leqslant t_1) \quad (j = 1, \ldots, r) \qquad (1)$$

In general, the solution of integral Equations (1) necessitates the application of numerical methods.

In connection with this, let us determine the transformation of the nonlinear integral Equations (1) to a system of finite transcendental equations, which can be obtained by approximating the desired functions $z_j(t)$ by stepwise functions. The solutions of these transcendental equations can be taken as the zeroth approximations in the construction of an iteration process for the solutions of the integral equations under consideration.

Let us divide the time interval (t_0, t_1) into ν equal or unequal intervals $(\alpha_{\mu-1}, \alpha_\mu)$, $(\mu = 1, \ldots, \nu)$, where $\alpha_0 = t_0$, $\alpha_\nu = t_1$, and, by assuming that the functions z_j are stepwise, let us denote by $z_{j\mu}$ the values taken by these functions in the intervals $(\alpha_{\mu-1}, \alpha_\mu)$

$$z_j(t) \equiv z_j(\alpha_{\mu-1}) = z_{j\mu} \quad (\alpha_{\mu-1} \leqslant t < \alpha_\mu) \tag{2}$$

By setting $t = \alpha_\mu - 0$, $(\mu = 1, \ldots, \nu)$, integral Equations (1) yield

$$z_{j\mu} = \Gamma_j(\alpha_\mu) - \sum_{i=0}^{m-1}\sum_{l=1}^{n} \chi_{ji}(\alpha_\mu)\left[\int_{t_0}^{\alpha_1} \Xi_{il}(t_1, \tau)\,\psi_l(z_{11}, \ldots, z_{r1}, \tau)\,d\tau + \int_{\alpha_1}^{\alpha_2} \Xi_{il}(t_1, \tau)\,\psi_l(z_{12}, \ldots, z_{r2}, \tau)\,d\tau + \ldots + \int_{\alpha_{\nu-1}}^{t_1} \Xi_{il}(t_1, \tau)\,\psi_l(z_{1\nu}, \ldots, z_{r\nu}, \tau)\,d\tau\right] + \sum_{l=1}^{n}\left[\int_{t_0}^{\alpha_1} W_{jl}(\alpha_\mu, \tau)\,\psi_l(z_{11}, \ldots, z_{r1}, \tau)\,d\tau + \int_{\alpha_1}^{\alpha_2} W_{jl}(\alpha_\mu, \tau)\,\psi_l(z_{12}, \ldots, z_{r2}, \tau)\,d\tau + \ldots + \int_{\alpha_{\mu-1}}^{\alpha_\mu} W_{jl}(\alpha_\mu, \tau)\,\psi_l(z_{1\mu}, \ldots, z_{r\mu}, \tau)\,d\tau\right] \quad (j = 1, \ldots, r;\ \mu = 1, \ldots, \nu) \tag{3}$$

With the notations

$$E_{j\mu\xi}(z_{1\xi}, \ldots, z_{r\xi}) = \sum_{i=0}^{m-1}\sum_{l=1}^{n} \chi_{ji}(\alpha_\mu) \int_{\alpha_{\xi-1}}^{\alpha_\xi} \Xi_{il}(t_1, \tau)\,\psi_l(z_{1\xi}, \ldots, z_{r\xi}, \tau)\,d\tau \tag{4}$$

$$L_{j\mu\xi}(z_{1\xi}, \ldots, z_{r\xi}) = \sum_{l=1}^{n} \int_{\alpha_{\xi-1}}^{\alpha_\xi} W_{jl}(\alpha_\mu, \tau)\,\psi_l(z_{1\xi}, \ldots, z_{r\xi}, \tau)\,d\tau \tag{5}$$

$$(j = 1, \ldots, r;\ \mu, \xi = 1, \ldots, \nu)$$

Equations (3) can be reduced to the following form:

$$\sum_{\xi=1}^{\nu} E_{j\mu\xi}(z_{1\xi}, \ldots, z_{r\xi}) - \sum_{\xi=1}^{\mu} L_{j\mu\xi}(z_{1\xi}, \ldots, z_{r\xi}) + z_{j\mu} = \Gamma_j(\alpha_\mu) \qquad (j = 1, \ldots, r;\ \mu = 1, \ldots, \nu) \tag{6}$$

Thus, we obtain a system of νr transcendental equations in the νr unknown quantities $z_{j\mu}$ $(j = 1, \ldots, r;\ \mu = 1, \ldots, \nu)$. The solutions of the system of Equations (6) in themselves determine stepwise functions which approximate the desired functions $z_j(t)$ $(j = 1, \ldots, r)$, in the interval $t_0 \leqslant t \leqslant t_1$.

In the special case when in the original system of differential equations of [1] there occurs only one nonlinear function ψ_λ depending on one argument z_k

$$\psi_\lambda = \psi_\lambda(z_k(t)) \tag{7}$$

then, in accordance with (1), we shall have a nonlinear integral equation in the unknown function $z_k(t)$

$$z_k(t) = \Gamma_k(t) - \sum_{i=0}^{m-1} \chi_{ki}(t) \int_{t_0}^{t_1} \Xi_{i\lambda}(t_1, \tau)\, \psi_\lambda(z_k(\tau))\, d\tau + \\ + \int_{t_0}^{t} W_{k\lambda}(t, \tau)\, \psi_\lambda(z_k(\tau))\, d\tau \quad (t_0 \leqslant t \leqslant t_1) \tag{8}$$

In the case considered here, Relations (3) take the form

$$z_{k\mu} = \Gamma_k(\alpha_\mu) - \sum_{i=0}^{m-1} \chi_{ki}(\alpha_\mu) \Big[\psi_\lambda(z_{k1}) \int_{t_0}^{\alpha_1} \Xi_{i\lambda}(t_1, \tau)\, d\tau + \psi_\lambda(z_{k2}) \int_{\alpha_1}^{\alpha_2} \Xi_{i\lambda}(t_1, \tau)\, d\tau + \ldots \\ \ldots + \psi_\lambda(z_{k\nu}) \int_{\alpha_{\nu-1}}^{t_1} \Xi_{i\lambda}(t_1, \tau)\, d\tau\Big] + \psi_\lambda(z_{k1}) \int_{t_0}^{\alpha_1} W_{k\lambda}(\alpha_\mu, \tau)\, d\tau + \\ + \psi_\lambda(z_{k2}) \int_{\alpha_1}^{\alpha_2} W_{k\lambda}(\alpha_\mu, \tau)\, d\tau + \ldots + \psi_\lambda(z_{k\mu}) \int_{\alpha_{\mu-1}}^{\alpha_\mu} W_{k\lambda}(\alpha_\mu, \tau)\, d\tau \quad (\mu = 1, \ldots, \nu) \tag{9}$$

Let us introduce the notations

$$e_{\mu\xi} = \sum_{i=0}^{m-1} \chi_{ki}(\alpha_\mu) \int_{\alpha_{\xi-1}}^{\alpha_\xi} \Xi_{i\lambda}(t_1, \tau)\, d\tau \qquad (\mu, \xi = 1, \ldots, \nu) \tag{10}$$

$$l_{\mu\xi} = \int_{\alpha_{\xi-1}}^{\alpha_\xi} W_{k\lambda}(\alpha_\mu, \tau)\, d\tau \tag{11}$$

Then system (9) can be reduced to the system of ν transcendental equations with constant coefficients in the ν unknown quantities z_{k1}, $\ldots$, $z_{k\nu}$:

$$\sum_{\xi=1}^{\nu} e_{\mu\xi}\psi_\lambda(z_{k\xi}) - \sum_{\xi=1}^{\mu} l_{\mu\xi}\psi_\lambda(z_{k\xi}) + z_{k\mu} = \Gamma_k(\alpha_\mu) \quad (\mu = 1, \ldots, \nu) \tag{12}$$

The solution of the system of Equations (12) determines a stepwise function which approximates the desired function $z_k(t)$ in the interval $t_0 \leqslant t \leqslant t_1$.

By denoting

$$\psi_\lambda (z_{k\xi}) = Z_{k\xi} \tag{13}$$

and by introducing the inverse function

$$z_{k\xi} = \zeta_\lambda (Z_{k\xi}) \tag{14}$$

we can transform the system of Equations (12) to the form

$$\sum_{\xi=1}^{\nu} e_{\mu\xi} Z_{k\xi} - \sum_{\xi=1}^{\mu} l_{\mu\xi} Z_k + \zeta_\lambda (Z_{k\mu}) = \Gamma_k (\alpha_\mu) \qquad (\mu = 1, \ldots, \nu) \tag{15}$$

which are simpler in comparison with system (12) since each of the equations in system (15) contains only one nonlinear component.

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